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Toda chains with type A_m Lie algebra for multidimensional m-component perfect fluid cosmology

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Abstract

We consider a D-dimensional cosmological model describing an evolution of Ricci-flat factor spaces, $M_1, \ldots, M_n$ ($n \geq 3$), in the presence of an m-component perfect fluid source ($n-1 \geq m \geq 2$). We find characteristic vectors, related to the matter constants in the barotropic equations of state for fluid components of all factor spaces. We show that, in the case where we can interpret these vectors as the root vectors of a Lie algebra of Cartan type $A_m = sl(m+1, \mathbf{C})$, the model reduces to the classical open m-body Toda chain. Using an elegant technique by Anderson [1] for solving this system, we integrate the Einstein equations for the model and present the metric in a Kasner-like form.

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1 Introduction

Recently, investigations on multidimensional gravitation and cosmology have found renewed interest. It is well known now, that multi-scalar-tensor models derived from a higher dimensional multidimensional Einstein action are similar to the (bosonic sector of) effective low-energy models from (super) string theory. Beyond that fact, it was recently shown that the multidimensional ansatz provides also a natural clue to membrane theory (as the natural generalization of string theory).

Here however, we restrict to multidimensional (spacially homogeneous) cosmology [28, 29]. The D-dimensional cosmological model describing the evolution of n ($n \geq 3$) Ricci-flat spaces $M_1, \ldots, M_n$ in the presence of an m-component ($n-1 \geq m \geq 2$) perfect fluid source is considered. The barotropic equations of state for the mass-energy densities and pressures of the components are given for each space. When the vectors related to the constants in the barotropic equations of state can be interpreted as root vectors of the Lie algebra $A_m = sl(m+1, \mathbf{C})$, the model reduces to the classical open-chain m-body Toda system. Using the new elegant form of its exact solution proposed in [1], we integrate the Einstein equations for the model and present the metric in the Kasner-like form.

2 The model and the equations of motion

The D-dimensional space-time manifold M may be the product of the time axis R and n manifolds $M_1, \ldots, M_n$, i.e.

$$M = R \times M_1 \times \ldots \times M_n. \tag{2.1}$$

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The product of some of the manifolds $M_1, \dots, M_k$, $1 \leq k \leq 3$, gives the external 3-dimensional space and the remaining part $M_{k+1}, \dots, M_n$ stands for so-called internal spaces. Further, for sake of generality, we admit that dimensions $N_i = \mathrm{dim} M_i$ for $i = 1, \dots, n$ are arbitrary.

The manifold M is equipped with the metric

$$\mathrm{d}s^2 = -\mathrm{e}^{2\gamma(t)}\mathrm{d}t^2 + \sum_{i=1}^{n} \exp[2x^i(t)]\mathrm{d}s_i^2, \tag{2.2}$$

where $\gamma(t)$ is an arbitrary function determining the time t and $\mathrm{d}s_i^2$ is the metric on the manifold M_i. We assume that the manifolds $M_1, \dots, M_n$ are Ricci-flat, i.e. the components of the Ricci tensor for the metrics $\mathrm{d}s_i^2$ are zero. Under this assumption the Ricci tensor for the metric (2.2) has following non-zero components

$$R_0^0 = \mathrm{e}^{-2\gamma}\left(\sum_{i=1}^{n} N_i(\dot{x}^i)^2 + \ddot{\gamma_0} - \dot{\gamma}\dot{\gamma_0}\right) \tag{2.3}$$

$$R_{n_i}^{m_i} = \mathrm{e}^{-2\gamma}\left[\ddot{x}^i + \dot{x}^i(\dot{\gamma_0} - \dot{\gamma})\right]\delta_{n_i}^{m_i} \tag{2.4}$$

with the definition

$$\gamma_0 = \sum_{i=1}^{n} N_i x^i. \tag{2.5}$$

Indices m_i and n_i in (2.3), (2.4) for $i = 1, \dots, n$ run from $(D - \sum_{j=i}^{n} N_j)$ to $(D - \sum_{j=i}^{n} N_j + N_i)$ $(D = 1 + \sum_{i=1}^{n} N_i = \mathrm{dim} M)$.

The source of gravity shall be a multicomponent perfect fluid. The energy-momentum tensor of such source comoving coordinates reads

$$T_N^M = \sum_{s=1}^{m} T_N^{M(s)}, \tag{2.6}$$

$$(T_N^{M(s)}) = \mathrm{diag}\left(-\rho^{(s)}(t), \underbrace{p_1^{(s)}(t), \dots, p_1^{(s)}(t)}_{N_1 times}, \dots, \underbrace{p_n^{(s)}(t), \dots, p_n^{(s)}(t)}_{N_n times}\right) \quad , \tag{2.7}$$

Furthermore we suppose that for any s-th component of the perfect fluid the barotropic equation of state is given by

$$p_i^{(s)}(t) = \left(1 - h_i^{(s)}\right)\rho^{(s)}(t), \quad s = 1, \dots, m, \tag{2.8}$$

where $h_i^{(s)} = \mathrm{const.}$

The equation of motion $\bigtriangledown_M T_0^{M(s)} = 0$ for the s-th component of the perfect fluid described by the tensor (2.7) reads

$$\dot{\rho}^{(s)} + \sum_{i=1}^{n} N_i\dot{x}^i\left(\rho^{(s)} + p_i^{(s)}\right) = 0. \tag{2.9}$$

Using the equations of state (2.8), we obtain from (2.9) the following integrals of motion

$$A^{(s)} = \rho^{(s)}\exp\left[2\gamma_0 - \sum_{i=1}^{n} N_i h_i^{(s)} x^i\right] = \mathrm{const.} \tag{2.10}$$

The Einstein equations $R_N^M - R\delta_N^M/2 = \kappa^2 T_N^M$ (κ^2 is the gravitational constant), can be written as $R_N^M = \kappa^2[T_N^M - T\delta_N^M/(D-2)]$. Furthermore, we employ the equations $R_0^0 - R/2 = \kappa^2 T_0^0$ and $R_{n_i}^{m_i} = \kappa^2[T_{n_i}^{m_i} - T\delta_{n_i}^{m_i}/(D-2)]$. Using (2.3)-(2.8), we obtain for them

$$\frac{1}{2}\sum_{i,j=1}^{n} G_{ij}\dot{x}^i\dot{x}^j + V = 0, \tag{2.11}$$

$$
\begin{aligned}
\ddot{x}^i + \dot{x}^i(\dot{\gamma_0} - \dot{\gamma}) &= -\kappa^2 \sum_{s=1}^{m} A^{(s)} \left(h_i^{(s)} - \frac{\sum_{k=1}^{n} N_k h_k^{(s)}}{D-2} \right) \\
&\times \ \exp\left[\sum_{i=1}^{n} N_i h_i^{(s)} x^i - 2(\gamma - \gamma_0) \right]. \qquad (2.12)
\end{aligned}
$$

Here

$$
G_{ij} = N_i \delta_{ij} - N_i N_j \tag{2.13}
$$

are the components of the minisuperspace metric,

$$
V = \kappa^2 \sum_{s=1}^{m} A^{(s)} \exp\left[\sum_{i=1}^{n} N_i h_i^{(s)} x^i - 2(\gamma - \gamma_0) \right]. \tag{2.14}
$$

(2.10) is used to replace the densities $\rho^{(s)}$ in (2.11), (2.12) by expressions of the functions $x^i(t)$.

After the gauge fixing $\gamma = F(x^1, \ldots, x^n)$ the equations of motion (2.12) are the Lagrange-Euler equations obtained from the Lagrangian

$$
L = \mathrm{e}^{\gamma_0 - \gamma} \left(\frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j - V \right) \tag{2.15}
$$

under the zero-energy constraint (2.12).

Now we introduce an n-dimensional real vector space $\mathbf{R^n}$. By $e_1, \ldots e_n$ we denote the canonical basis in $\mathbf{R^n}$ ($e_1 = (1, 0, \ldots, 0)$ etc.). Hereafter we use the following vectors:
1. the vector x with components being the solution of the equations of motion

$$
x = x^1(t) e_1 + \ldots + x^n(t) e_n, \tag{2.16}
$$

2. m vectors u_s, each of them one component of the perfect fluid

$$
u_s = \sum_{i=1}^{n} \left(h_i^{(s)} - \frac{\sum_{k=1}^{n} N_k h_k^{(s)}}{D-2} \right) e_i. \tag{2.17}
$$

Let $< .,. >$ be a symmetrical bilinear form defined on $\mathbf{R^n}$ such that

$$
< e_i, e_j >= G_{ij}. \tag{2.18}
$$

The form is nongenerated and the inverse matrix to (G_{ij}) has the components

$$
G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2-D}. \tag{2.19}
$$

The form $< .,. >$ endows the space $\mathbf{R^n}$ with a metric, the signature of which is $(-, +, ..., +)$ [20],[21]. G_{ij} is used to introduce the covariant components of vectors u_s

$$
u_i^{(s)} = \sum_{i=1}^{n} G_{ij} u^j_{(s)} = N_i h_i^{(s)}. \tag{2.20}
$$

Form them the bilinear form reads

$$
< u_s, u_r >= \sum_{i=1}^{n} h_i^{(s)} h_i^{(r)} N_i + \frac{1}{2-D} \left[\sum_{i=1}^{n} h_i^{(s)} N_i \right] \left[\sum_{j=1}^{n} h_j^{(r)} N_j \right]. \tag{2.21}
$$

A vector $y \in \mathbf{R^n}$ is called time-like, space-like or isotropic, if $< y, y >$ is smaller, greater than or equal to zero, correspondingly. The vectors y and z are called orthogonal if $< y, z >= 0$.

Using the notation $< ., . >$ and the vectors (2.16)-(2.17), we may write the zero-energy constraint (2.11) and the Lagrangian (2.15) in the form

$$E = \frac{1}{2} < \dot{x}, \dot{x} > + \kappa^2 \mathrm{e}^{2(\gamma-\gamma_0)} \sum_{s=1}^{m} A^{(s)} \mathrm{e}^{<u_s,x>} = 0, \tag{2.22}$$

$$L = \frac{1}{2} \mathrm{e}^{\gamma_0-\gamma} < \dot{x}, \dot{x} > - \kappa^2 \mathrm{e}^{\gamma-\gamma_0} \sum_{s=1}^{m} A^{(s)} \mathrm{e}^{<u_s,x>}. \tag{2.23}$$

Furthermore, we take the so called harmonic time gauge, which implies

$$\gamma(t) = \gamma_0 = \sum_{i=1}^{n} N_i x^i. \tag{2.24}$$

¿From the mathematical point of view the problem consist in solving the dynamical system, described by a Lagrangian of the general form

$$L = \frac{1}{2} < \dot{x}, \dot{x} > - \sum_{s=1}^{m} a^{(s)} \mathrm{e}^{<u_s,x>}, \tag{2.25}$$

where $x, u_s \in \mathbf{R^n}$. It should be noted that the kinetic term $< \dot{x}, \dot{x} >$ is not a positively definite bilinear form as it is usually the case in classical mechanics. Due to the pseudo-Euclidean signature $(-, +, ..., +)$ of the form $< ., . >$ such systems may be called pseudo-Euclidean Toda-like systems as the potential given in (2.25) defines a well known in classical mechanics Toda lattices [35].

Note that, we have to integrate the equations of motion following from the Lagrangian (2.25) under the zero-energy constraint. Although an additional constant term $-a^{(0)}$ (with $u_0 \equiv 0 \in \mathbf{R^n}$) in the Lagrangian (2.25) does not change the equations of motion, it nevertheless shifts the energy constraint from zero to

$$E \equiv \frac{1}{2} < \dot{x}, \dot{x} > + \sum_{s=1}^{m} a^{(s)} \exp[< u_s, x >] = -a^{(0)} \equiv -\kappa^2 A^{(0)}. \tag{2.26}$$

In our cosmological model, with (2.8) and (2.17), such a term corresponds to a perfect fluid with $h_i^{(0)} = 0$ for all $i = 1, \ldots, n$. This is in fact just a Zeldovich (stiff) matter component, which can also be interpreted as a minimally coupled real scalar field. Taking into account the possible presence of Zeldovich matter, we have now to integrate the equations of motion for an arbitrary energy level E.

3 Solving the equations of motion for the model which reduces to a classical open Toda chain

We start from the Lagrangian (2.25) and the energy constraint (2.26) with

$$n \geq m + 1 \quad , \quad m \geq 2 \quad . \tag{3.1}$$

The vectors u_s may obey the relations

$$\begin{aligned} < u_s, u_s > &= u^2 > 0 \quad , \quad s = 1, \ldots, m \quad , & (3.2)\\ < u_r, u_{r+1} > &= -\frac{1}{2} u^2 \quad , \quad , r = 1, \ldots, m-1 \quad , & (3.3)\\ \text{all the remaining} \quad < u_r, u_s > &= 0 \quad , & (3.4) \end{aligned}$$

where u is an arbitrary non-zero real number. In this case the vectors u_s are space-like, linearly independent, and can be interpreted as root vectors of the Lie algebra

$A_m = sl(m+1, \mathbf{C})$. The Cartan matrix (K_{rs}) (see e.g. [17, 18]) then reads

$$(K_{rs}) = \left(\frac{2 < u_r, u_s >}{< u_r, u_r >} \right) = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix} \quad . \tag{3.5}$$

Now, we choose in $\mathbf{R^n}$ a basis $\{f_1, \dots, f_n\}$ with the following properties

$$f_{s+1} = \frac{2u_s}{< u_s, u_s >} \quad , \quad s = 1, \dots, m \quad , \tag{3.6}$$

$$< f_1, f_i > = \eta_{1i} \quad , \quad i = 1, \dots, n \quad , \tag{3.7}$$

$$< f_{s+1}, f_k > = 0 \, , \; < f_k, f_l >= \eta_{kl} \quad , \quad s = 1, \dots, m \quad ; \quad k, l = m+2, \dots, n \tag{3.8}$$

with

$$(\eta_{ij}) = diag(-1, +1, \dots, +1) \quad , \quad i, j = 1, \dots, n \quad . \tag{3.9}$$

Note that, the basis contains, besides vectors proportional to u_s, additional vectors $f_{m+2}, \dots, f_n$, iff $n > m+1$. By the decomposition

$$x(t) = \sum_{i=1}^{n} q^i(t) f_i \tag{3.10}$$

w.r.t. this basis, with relations (3.2) - (3.4), (3.6) - (3.8) the Lagrangian (2.25) takes the form

$$\begin{aligned} L \;=\; & \frac{1}{2} \left(- \left(\dot{q}^1 \right)^2 + \frac{4}{u^2} \left[\sum_{s=2}^{m+1} (\dot{q}^s)^2 - \sum_{p=2}^{m} \dot{q}^p \dot{q}^{p+1} \right] + \sum_{k=m+2}^{n} \left(\dot{q}^k \right)^2 \right) \\ & - a^{(1)} e^{2q^2 - q^3} - \sum_{r=3}^{m} a^{(r-1)} e^{2q^r - q^{r-1} - q^{r+1}} - a^{(m)} e^{2q^{m+1} - q^m} \quad . \end{aligned} \tag{3.11}$$

The equations of motion for $q^1(t)$, $q^{m+2}(t)$,...,$q^n(t)$ read

$$\ddot{q}^1(t) = 0 \, , \; \ddot{q}^{m+2} = 0 \, , \; \dots, \; \ddot{q}^n(t) = 0 \, . \tag{3.12}$$

Then,

$$q^1(t) = a^1 t + b^1 \tag{3.13}$$

$$q^k(t) = a^k t + b^k \, , \; k = m+2, \dots, n. \tag{3.14}$$

The other equations of motion for $q^2(t)$,...,$q^{m+1}(t)$ follow from the Lagrangian

$$L_E = \sum_{s=2}^{m+1} (\dot{q}^s)^2 - \sum_{p=2}^{m} \dot{q}^p \dot{q}^{p+1} - \frac{u^2}{2} \left[a^{(1)} e^{2q^2 - q^3} + \sum_{r=3}^{m} a^{(r-1)} e^{2q^r - q^{r-1} - q^{r+1}} + a^{(m)} e^{2q^{m+1} - q^m} \right] \quad . \tag{3.15}$$

The linear transformation

$$q^{s+1} \longrightarrow q^s - \ln C^s \, , \; s = 1, \dots, m \, , \tag{3.16}$$

where the constants C^1,...,C^m have to satisfy

$$\sum_{s=1}^{m} K_{rs} \ln C^s = \ln \frac{u^2 a^{(r)}}{2}, \; r = 1, \dots, m \, , \tag{3.17}$$

brings the Lagrangian into the form

$$L_{A_m} = \sum_{s=1}^{m} (\dot{q}^s)^2 - \sum_{r=1}^{m-1} \dot{q}^r \dot{q}^{r+1} - e^{2q^1 - q^2} - \sum_{p=2}^{m-1} e^{2q^p - q^{p-1} - q^{p+1}} - e^{2q^m - q^{m-1}} \, . \tag{3.18}$$

The latter represents the Lagrangian of a Toda chain associated with the Lie algebra A_m [35] when the root vectors are put into the Chevalley basis and coordinates describing the motion of the mass center are separated out.

We use the method suggested in [1] for solving the equations of motion following from (3.18) and obtain

$$e^{-q^s} \equiv F_s(t) = \sum_{r_1<\ldots<r_s}^{m+1} v_{r_1} \cdots v_{r_s} \Delta^2(r_1, \ldots, r_s) e^{(w_{r_1}+\ldots+w_{r_s})t} \tag{3.19}$$

where $\Delta^2(r_1, \ldots, r_s)$ denotes the square of the Vandermonde determinant

$$\Delta^2(r_1, \ldots, r_s) = \prod_{r_i<r_j} \left(w_{r_i} - w_{r_j}\right)^2 \ . \tag{3.20}$$

The constants v_r and w_r have to satisfy the relations

$$\prod_{r=1}^{m+1} v_r = \Delta^{-2}(1, \ldots, m+1) \ , \tag{3.21}$$

$$\sum_{r=1}^{m+1} w_r = 0 \ . \tag{3.22}$$

The energy of the Toda chain described by this solution is given by

$$E_0 = \frac{1}{2} \sum_{r=1}^{m+1} w_r^2 \ . \tag{3.23}$$

Finally, we obtain the folowing decomposition of the vector $x(t)$

$$x(t) = (a^1 t + b^1) f_1 + \sum_{s=1}^{m} \frac{-2\left(\ln F_s(t) + \ln C^s\right)}{< u_s, u_s >} u_s + \sum_{k=m+2}^{m} (a^k t + b^k) f_k \tag{3.24}$$

We remind the reader that the coordinates $x^i(t)$ of the vector $x(t)$ are, with respect to the *canonical* basis in $\mathbf{R^n}$, the logarithms of the scale factors in the corresponding cosmological model.

Let us introduce the vectors

$$\alpha = a^1 f_1 + \sum_{k=m+2}^{n} a^k f_k \equiv \sum_{i=1}^{n} \alpha^i e_i \in \mathbf{R}^n \tag{3.25}$$

$$\beta = b^1 f_1 + \sum_{k=m+2}^{n} b^k f_k \equiv \sum_{i=1}^{n} \beta^i e_i \in \mathbf{R}^n \tag{3.26}$$

with α^i, β^i being their coordinates with respect to the canonical basis. Using (3.7) and (3.8), we conclude these coordinates have to satisfy the following equations

$$< \alpha, u_s >= \sum_{i,j=1}^{n} G_{ij} \alpha^i u^j_{(s)} = 0 \ , \ s = 1, \ldots, m \ , \tag{3.27}$$

$$< \beta, u_s >= \sum_{i,j=1}^{n} G_{ij} \beta^i u^j_{(s)} = 0 \ , \ s = 1, \ldots, m \ , \tag{3.28}$$

where the $u^i_{(s)}$ are the coordinates of u_s in the canonical basis (see (2.17)).

The total energy E of the system is given by

$$E = \frac{1}{2} < \alpha, \alpha > + \frac{2}{u^2} E_0 = \frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \alpha^i \alpha^j + \frac{1}{u^2} \sum_{s=1}^{m+1} (w_s)^2 \ . \tag{3.29}$$

If $m=n+1$, then $<\alpha,\alpha>=-\left(a^1\right)^2\leq 0$. With (3.29), we then obtain $E\leq\frac{2}{u^2}E_0$.

Finally, the scale factors of the multidimensional cosmological model with the Lagrangian (2.25) and the energy constraint (2.26) are given by

$$e^{x^i(t)}=\prod_{s=1}^{m}\left[\tilde{F}_s^2(t)\right]^{-u^i_{(s)}/<u_s,u_s>}e^{\alpha^i t+\beta^i} \quad , \tag{3.30}$$

where

$$\tilde{F}_s(t)=C^s\cdot F_s(t)\ ,\ s=1,\ldots,m\ . \tag{3.31}$$

Using (2.10) we obtain the following solution for the densities of the perfect fluid components

$$\begin{aligned}\rho^{(1)}&=A^{(1)}\mathrm{e}^{-2\gamma_0}\frac{\tilde{F}_2}{\tilde{F}_1^2}\ ,\quad \rho^{(m)}=A^{(m)}\mathrm{e}^{-2\gamma_0}\frac{\tilde{F}_{m-1}}{\tilde{F}_m^2}\\ \rho^{(p)}&=A^{(p)}\mathrm{e}^{-2\gamma_0}\frac{\tilde{F}_{p-1}\tilde{F}_{p+1}}{\tilde{F}_p^2}\ ,\quad p=2,\ldots,m-1.\end{aligned} \tag{3.32}$$

where γ_0 is defined by (2.5) and may be calculated by (3.30).

The constants C^s are specified by (3.17). The solution contains the parameters $\alpha^i,\ \beta^i,\ v_r,\ w_r$ ($i=1,\ldots,n,\ r=1,\ldots,m+1$) obeying the constraints (3.27),(3.28),(3.21),(3.22),(3.29). If the energy E is arbitrary (see (2.26)) the solution has $2n$ free parameters as required.

4 Example

We consider a space-time manifold M with the following structure

$$M=R\times M_1^3\times M_2^3\times M_3^4 \tag{4.1}$$

where the dimensions of M_1^3, M_2^3, and M_3^4 are 3, 3, and 4, respectively. The first component of the perfect fluid shall have the $h_i^{(1)}$ values

$$h_1^{(1)}=0\quad,\quad h_2^{(1)}=h\quad,\quad h_3^{(1)}=0 \tag{4.2}$$

while the second component is given by

$$h_1^{(2)}=h\quad,\quad h_2^{(2)}=0\quad,\quad h_3^{(2)}=0\quad . \tag{4.3}$$

h is a real valued parameter with the restriction

$$h\neq 0. \tag{4.4}$$

It is easy to check that the relations (3.2), (3.3) with $s=2$ and $r=1$ are fulfilled.

In this case, the exact solution of the field equations gives the metric

$$\begin{aligned}ds^2 \;=\;& \left[\tilde{F}_1(t)\tilde{F}_2(t)\right]^{\frac{2}{3h}}\\ &\times\left\{-\exp(8\alpha_0 t+8\beta_0)dt^2+\frac{ds_1^2}{\tilde{F}_2^{\frac{2}{h}}(t)}+\frac{ds_2^2}{\tilde{F}_1^{\frac{2}{h}}(t)}+\exp(2\alpha_0 t+2\beta_0)ds_3^2\right\}\end{aligned} \tag{4.5}$$

with the following definitions

$$\tilde{F}_1(t)=\kappa^2\left[A^{(1)}\right]^{\frac{2}{3}}\left[A^{(2)}\right]^{\frac{1}{3}}h^2F_1(t)\quad , \tag{4.6}$$

and

$$\tilde{F}_2(t)=\kappa^2\left[A^{(1)}\right]^{\frac{1}{3}}\left[A^{(2)}\right]^{\frac{2}{3}}h^2F_2(t)\quad , \tag{4.7}$$

and

$$F_1(t) = v_1 e^{w_1 t} + v_2 e^{w_2 t} + v_3 e^{w_3 t} \quad , \tag{4.8}$$

$$\begin{aligned} F_2(t) \quad &= \quad v_1 v_2 (w_1 - w_2)^2 e^{(w_1+w_2)t} + v_1 v_3 (w_1 - w_3)^2 e^{(w_1+w_3)t} \\ &\quad + v_2 v_3 (w_2 - w_3)^2 e^{(w_2+w_3)t} \quad . \end{aligned} \tag{4.9}$$

In our case, the energy E is given by

$$E = -6\alpha_0^2 + \frac{1}{2h^2}\left[w_1^2 + w_2^2 + w_3^2\right] = -\kappa^2 A^{(0)} \quad . \tag{4.10}$$

$A^{(0)} > 0$ means that Zeldovich matter is present in all the internal spaces (See the remarks at the end of sect. 3). With $A^{(0)} = 0$ (4.10) is the energy constraint specialized to our example.

The nine parameters $w_1, w_2, w_3, v_1, v_2, v_3, \alpha_0, \beta_0, E$ have to satisfy (4.9) and the two further relations

$$w_1 + w_2 + w_3 = 0 \tag{4.11}$$

and

$$v_1 v_2 v_3 = \left[(w_1 - w_2)(w_2 - w_3)(w_1 - w_3)\right]^{-2} \tag{4.12}$$

(See (3.22) and (3.21)!).

Finally, we have to give the expressions for the matter densities $\rho^{(1)}$ and $\rho^{(2)}$. They read

$$\rho^{(1)} = A^{(1)}\left[\tilde{F}_1^{-2/(3h)-2}(t)\tilde{F}_2^{1-2/(3h)}(t)\right] e^{-8\alpha_0 t - 8\beta_0} \quad , \tag{4.13}$$

$$\rho^{(2)} = A^{(2)}\left[\tilde{F}_2^{-2/(3h)-2}(t)\tilde{F}_1^{1-2/(3h)}(t)\right] e^{-8\alpha_0 t - 8\beta_0} \tag{4.14}$$

and their quotient is

$$\frac{\rho^{(2)}}{\rho^{(1)}} = \frac{A^{(2)}}{A^{(1)}}\frac{\tilde{F}_1(t)}{\tilde{F}_2(t)} \quad . \tag{4.15}$$

Although the solution is invariant with respect to the exchange $(w_1, w_2, v_1, v_2) \to (w_2, w_1, v_2, v_1)$ there is still enough freedom to build a lot of solutions that it is difficult to recognize many general properties of the solutions. What one can say is the following: We know that

$$\left(e^{x^1(t)}\right)^{3h} \propto \left|\frac{F_1(t)}{F_2^2(t)}\right| \tag{4.16}$$

and

$$\left(e^{x^2(t)}\right)^{3h} \propto \left|\frac{F_2(t)}{F_1^2(t)}\right| \quad . \tag{4.17}$$

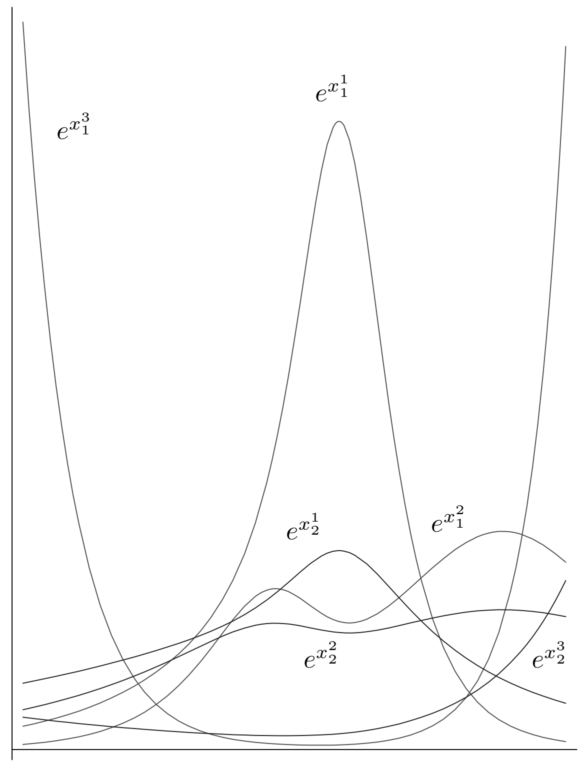
An easy but tedious discussion of the different possibilities of choosing the parameters w_1 and w_2 shows that the expressions (4.16) and (4.17) have for $t \to \pm\infty$ the following shape:

$$\left|\frac{F_2(t)}{F_1^2(t)}\right| \quad \stackrel{t\to\pm\infty}{\longrightarrow} \quad e^{f_{\pm\infty}(w_1,w_2)t} \tag{4.18}$$

and

$$\left|\frac{F_1(t)}{F_2^2(t)}\right| \quad \stackrel{t\to\pm\infty}{\longrightarrow} \quad e^{g_{\pm\infty}(w_1,w_2)t} \tag{4.19}$$

with some functions $f_{\pm\infty}$, and $g_{\pm\infty}$ of the parameters w_1 and w_2 being negative for $+\infty$ and positive for $-\infty$. This shows that the scale factors of the manifold M_1 and M_2 go to zero for $t \to \pm\infty$.Because of the behaviour of the scale factor of M_1, as one could have expected, the solution can not properly describe the total cosmic evolution. But what one can do is arranging especially the parameters w_1 and w_2 such that the behaviour of the scale factors is acceptable from a physical point of view (see below).



For the parameter values $v_1 = 1$, $v_2 = 10$, $w_1 = 1.1$, $w_2 = 5$ and two different values of h ($h_1 = 2/3$, $h_2 = 2$) (Zeldovich matter is present) the scale factors $x^i(t)$ are pictured on a certain time intervall. x_k^i is x^i for $h = h_k$. The integration constants $A^{(s)}$ are put equal to 1.

As for the scale factor $e^{x^3(t)}$ of the manifold M_3, we have

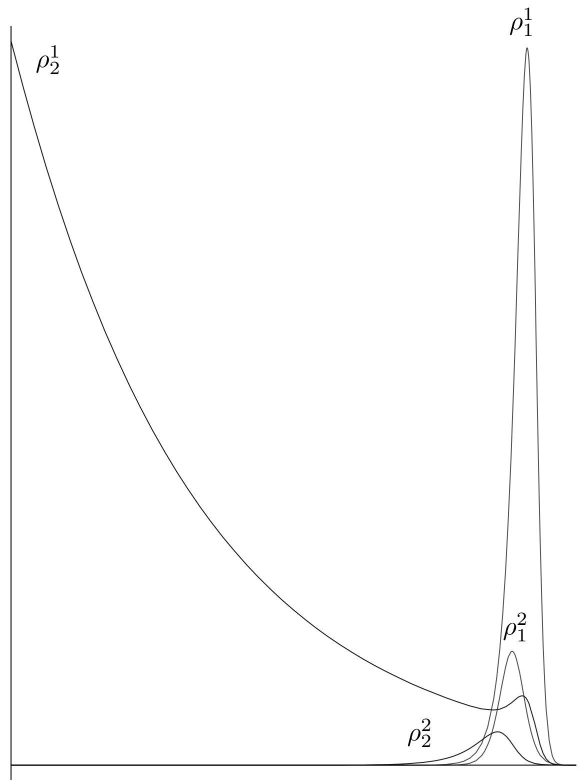
$$\left(e^{x^3(t)}\right)^{3h} \propto |F_1 F_2| \, e^{3\alpha_0 h t} \tag{4.20}$$

(the parameter β_0 shall be absorbed by coordinates). For $t \to \pm\infty$ we have

$$|F_1 F_2| \quad \to \quad e^{\pm k(w_1, w_2) t} \tag{4.21}$$

where $k(w_1, w_2)$ is some positive function of the parameters w_1 and w_2.

The proper time T as a function of harmonic time t is given by integration of $dT = e^{\gamma_0} dt$ with $\gamma_0 = 3x^1 + 3x^2 + 4x^3$. For $t \to \pm\infty$ the behaviour of both, the proper time T and the scale factor e^{x^3} depends on the choice of α_0. This holds for the case that in all manifolds Zeldovich matter is present. If the case is excluded then α_0 is given by the energy constraint. We shall not discuss this point here because of the above mentioned restricted range of possibly physical importance of the solution. Instead, we present some drawings conveying how the scale factors and matter densities can behave for special values of w_1, w_2, v_1, v_2, and α_0.



The energy densities ρ_1 and ρ_2 belonging to the parameter values $v_1 = 1$, $v_2 = 10$, $w_1 = 1.1$, $w_2 = 5$ and two different values of h ($h_1 = 2/3$, $h_2 = 2$) (Zeldovich matter is present). ρ_i^k means the energy density ρ_i belonging to h_k.

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